

The Lusztig automorphism of $U_q(\mathfrak{sl}_2)$ from the equitable point of view

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Abstract

We consider the quantum algebra $U_q(\mathfrak{sl}_2)$ in the equitable presentation. From this point of view, we describe the Lusztig automorphism and the corresponding Lusztig operator.

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1 Introduction

This paper is about the quantum algebra $U_q(\mathfrak{sl}_2)$, but in order to motivate things we have some preliminary comments about the Lie algebra \mathfrak{sl}_2 . Let \mathbb{F} denote a field with characteristic zero, and consider the Lie algebra \mathfrak{sl}_2 over \mathbb{F} . The following facts are taken from [5, Sections 2.3, 7]. The Lie algebra \mathfrak{sl}_2 has a basis e, f, h and Lie bracket

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

On each finite-dimensional \mathfrak{sl}_2 -module, h is diagonalizable and e, f are nilpotent. The Lie algebra \mathfrak{sl}_2 has an automorphism

$$\mathcal{L} = \exp(\operatorname{ad} e) \exp(-\operatorname{ad} f) \exp(\operatorname{ad} e), \quad (1)$$

where $\operatorname{ad} u(v) = [u, v]$ and $\exp(\varphi) = \sum_{i \in \mathbb{N}} \varphi^i / i!$. The automorphism \mathcal{L} sends

$$e \mapsto -f, \quad f \mapsto -e, \quad h \mapsto -h.$$

The operator

$$\mathcal{T} = \exp(e) \exp(-f) \exp(e) \quad (2)$$

acts on nonzero finite-dimensional \mathfrak{sl}_2 -modules. On these modules,

$$\mathcal{L}(\xi) = \mathcal{T} \xi \mathcal{T}^{-1} \quad \forall \xi \in \mathfrak{sl}_2.$$

We are done discussing \mathfrak{sl}_2 . We now turn our attention to the analog situation for $U_q(\mathfrak{sl}_2)$. From now on, let the field \mathbb{F} be arbitrary. Fix a nonzero $q \in \mathbb{F}$ that is not a root of 1, and

consider the algebra $U_q(\mathfrak{sl}_2)$ over \mathbb{F} . By [9, Definition 1.1], the Chevalley presentation of $U_q(\mathfrak{sl}_2)$ has generators $e, f, k^{\pm 1}$ and relations $kk^{-1} = 1, k^{-1}k = 1$,

$$ke = q^2ek, \quad kf = q^{-2}fk, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

The Lusztig automorphism L of $U_q(\mathfrak{sl}_2)$ was introduced in [10, 11] as a quantum analog of (1). By [9, Section 8.14], L sends

$$e \mapsto -fk, \quad f \mapsto -k^{-1}e, \quad k \mapsto k^{-1}.$$

The Lusztig operator T was introduced in [10, 11] as a quantum analog of (2). T acts on a family of finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules, said to have type 1. This family is described as follows. A nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -module V has type 1 if and only if k is diagonalizable on V , with all eigenvalues among $\{q^\lambda\}_{\lambda \in \mathbb{Z}}$. Assume that V has type 1. For $\lambda \in \mathbb{Z}$ the corresponding weight space $V(\lambda) = \{v \in V | kv = q^\lambda v\}$. By [9, Section 8.2] we find that on each weight space $V(\lambda)$,

$$T = \sum_{\substack{a, b, c \in \mathbb{N} \\ b-a-c=\lambda}} \frac{e^a f^b e^c}{[a]_q! [b]_q! [c]_q!} (-1)^b q^{b-ac}.$$

(The bracket notation is explained in Section 2.) By [9, Lemmas 8.4, 8.5], on V the operator T is invertible and

$$L(\xi) = T\xi T^{-1} \quad \forall \xi \in U_q(\mathfrak{sl}_2).$$

The equitable presentation of $U_q(\mathfrak{sl}_2)$ was introduced in [8] and investigated further in [1, 3, 4, 6, 7, 12, 14–19]. This presentation has generators $x, y^{\pm 1}, z$ and relations $yy^{-1} = 1, y^{-1}y = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

In Section 5 we explain how x, y, z are related to e, f, k . This relationship is not unique. From the set of possible relationships we select two that seem attractive (see Corollary 5.17); these exist provided that $q^{1/2} \in \mathbb{F}$. For the rest of this section, assume that $q^{1/2} \in \mathbb{F}$ and adopt one of the two selections.

Our goal for this paper is to describe how L and T look in the equitable presentation. We now describe L . Define

$$n_x = \frac{q(1 - yz)}{q - q^{-1}}, \quad n_y = \frac{q(1 - zx)}{q - q^{-1}}, \quad n_z = \frac{q(1 - xy)}{q - q^{-1}}.$$

By [8, Lemma 5.4],

$$\begin{aligned} xn_y &= q^2 n_y x, & yn_z &= q^2 n_z y, & zn_x &= q^2 n_x z, \\ xn_z &= q^{-2} n_z x, & yn_x &= q^{-2} n_x y, & zn_y &= q^{-2} n_y z. \end{aligned}$$

On nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules, each of x, y, z is invertible (see Lemma 5.15) and each of n_x, n_y, n_z is nilpotent (see Lemma 5.14.) By [15, Lemma 6.4], the algebra $U_q(\mathfrak{sl}_2)$ is generated by $n_x, y^{\pm 1}, n_z$. As we will see in Corollary 5.17, L sends

$$n_x \mapsto y^{-1} n_z y^{-1}, \quad y \mapsto y^{-1}, \quad n_z \mapsto n_x.$$

We now describe T . We define an operator Υ that acts on finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules of type 1. Let V denote such a module. On each weight space $V(\lambda)$, Υ acts as $q^{-\lambda^2/2}$ times the identity. We recall the notion of a rotator. This notion is implicit in [8] and explicit in [16, Section 16]; see also [18, Section 22]. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. A rotator on V is an invertible $R \in \text{End}(V)$ such that on V ,

$$R^{-1}xR = y, \quad R^{-1}yR = z, \quad R^{-1}zR = x.$$

Recall the q -exponential function

$$\exp_q(\varphi) = \sum_{i \in \mathbb{N}} \frac{q^{\binom{i}{2}}}{[i]_q!} \varphi^i.$$

As we will see in Lemma 9.8, the operator

$$\mathfrak{R} = \exp_q(n_x) \Upsilon \exp_q(n_z)$$

acts as a rotator on each finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. We can now easily describe T in the equitable presentation. In Theorem 9.9 we show that on each finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1,

$$T^{-1} = \exp_q(n_z) \mathfrak{R}.$$

We have been describing the Lusztig automorphism L and the Lusztig operator T . For both maps there is a second version, which we denote by L^\vee and T^\vee , respectively. The maps L^\vee, T^\vee are treated along with L, T in the main body of the paper.

The paper is organized as follows. Section 2 contains some preliminaries. Section 3 contains some basic facts about the Chevalley presentation of $U_q(\mathfrak{sl}_2)$. In Section 4 we discuss how the Lusztig automorphisms L, L^\vee and the Lusztig operators T, T^\vee look in the Chevalley presentation. In Section 5 we discuss the equitable presentation of $U_q(\mathfrak{sl}_2)$, and describe how L, L^\vee look in this presentation. In Section 6 we review the q -exponential function, and apply it to n_x, n_y, n_z . Section 7 is about rotators. In Sections 8, 9 we use rotators to describe how the Lusztig operators T, T^\vee look in the equitable presentation. Theorem 9.9 is the main result of the paper.

2 Preliminaries

We now begin our formal argument. Throughout the paper the following notation and assumptions are in effect. An algebra without the Lie prefix is meant to be associative and have a 1. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$.

Let \mathbb{F} denote a field, and let V denote a vector space over \mathbb{F} with finite positive dimension. Let $\text{End}(V)$ denote the \mathbb{F} -algebra consisting of the \mathbb{F} -linear maps from V to V . An element $\varphi \in \text{End}(V)$ is called *diagonalizable* whenever V is spanned by the eigenspaces of φ . The map φ is called *multiplicity-free* whenever φ is diagonalizable, and each eigenspace has dimension one. The map φ is called *nilpotent* whenever there exists a positive integer n such that $\varphi^n = 0$. Fix a nonzero $q \in \mathbb{F}$ that is not a root of 1. For $n \in \mathbb{Z}$ define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

and for $n \geq 0$ define

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q.$$

We interpret $[0]_q! = 1$.

3 The Chevalley presentation of $U_q(\mathfrak{sl}_2)$

We recall the quantum algebra $U_q(\mathfrak{sl}_2)$ in the Chevalley presentation, following the treatment in [9].

Definition 3.1. (See [9, Definition 1.1].) Let $U_q(\mathfrak{sl}_2)$ denote the \mathbb{F} -algebra with generators $e, f, k^{\pm 1}$ and relations

$$\begin{aligned} kk^{-1} &= 1, & k^{-1}k &= 1, \\ ke &= q^2ek, & kf &= q^{-2}fk, \\ ef - fe &= \frac{k - k^{-1}}{q - q^{-1}}. \end{aligned}$$

The elements $e, f, k^{\pm 1}$ are called the *Chevalley generators* for $U_q(\mathfrak{sl}_2)$.

Define

$$\Lambda = (q - q^{-1})^2 ef + q^{-1}k + qk^{-1}. \quad (3)$$

We call Λ the *normalized Casimir element* of $U_q(\mathfrak{sl}_2)$. The element $\Lambda(q - q^{-1})^{-2}$ is the Casimir element of $U_q(\mathfrak{sl}_2)$ discussed in [9, Section 2.7]. By [9, Proposition 2.18] the elements $\{\Lambda^n\}_{n \in \mathbb{N}}$ form a basis for the center of $U_q(\mathfrak{sl}_2)$.

We now recall the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -modules.

Lemma 3.2. (See [9, Theorem 2.6].) *There exists a family of finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -modules*

$$\mathbf{V}_{d,\varepsilon} \quad d \in \mathbb{N}, \quad \varepsilon \in \{1, -1\} \quad (4)$$

with the following properties: $\mathbf{V}_{d,\varepsilon}$ has a basis $\{v_i\}_{i=0}^d$ such that

$$\begin{aligned} kv_i &= \varepsilon q^{d-2i} v_i & (0 \leq i \leq d), \\ fv_i &= [i+1]_q v_{i+1} & (0 \leq i \leq d-1), \quad fv_d = 0, \\ ev_i &= \varepsilon [d-i+1]_q v_{i-1} & (1 \leq i \leq d), \quad ev_0 = 0. \end{aligned}$$

Each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module is isomorphic to exactly one of the modules from line (4).

We have some comments about Lemma 3.2. The $U_q(\mathfrak{sl}_2)$ -module $\mathbf{V}_{d,\varepsilon}$ has dimension $d+1$. Referring to line (4), if $\text{Char}(\mathbb{F}) = 2$ then we view $\{1, -1\}$ as containing a single element. On the $U_q(\mathfrak{sl}_2)$ -module $\mathbf{V}_{d,\varepsilon}$ the element k is multiplicity-free, with eigenvalues $\{\varepsilon q^{d-2i} \mid 0 \leq i \leq d\}$. Moreover each of e^r, f^r is nonzero for $0 \leq r \leq d$, and

$$e^{d+1} = 0, \quad f^{d+1} = 0.$$

By [9, Lemma 2.7] the normalized Casimir element Λ acts on $\mathbf{V}_{d,\varepsilon}$ as $\varepsilon(q^{d+1} + q^{-d-1})I$.

Definition 3.3. Referring to the $U_q(\mathfrak{sl}_2)$ -module $\mathbf{V}_{d,\varepsilon}$ from Lemma 3.2, we call d and ε the *diameter* and *type*, respectively. We often abbreviate $\mathbf{V}_d = \mathbf{V}_{d,1}$.

Next we consider finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules that are not necessarily irreducible.

Lemma 3.4. (See [9, Proposition 2.1].) *The elements e and f are nilpotent on every finite-dimensional $U_q(\mathfrak{sl}_2)$ -module.*

Lemma 3.5. (See [9, Proposition 2.3, Theorem 2.9].) *Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module. Then the following are equivalent:*

- (i) *the action of k on V is diagonalizable;*
- (ii) *V is a direct sum of irreducible $U_q(\mathfrak{sl}_2)$ -submodules.*

Moreover, if $\text{Char}(\mathbb{F}) \neq 2$ then the conditions (i), (ii) hold.

Definition 3.6. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module. Then V is said to have *type 1* whenever $V \neq 0$ and V is a direct sum of irreducible $U_q(\mathfrak{sl}_2)$ -submodules that have type 1.

From our above comments we routinely obtain the following result.

Lemma 3.7. *Let V denote a nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -module. Then the following are equivalent:*

- (i) *V has type 1;*
- (ii) *the action of k on V is diagonalizable, and each eigenvalue is among $\{q^\lambda\}_{\lambda \in \mathbb{Z}}$.*

Lemma 3.8. *Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. Then each nonzero $U_q(\mathfrak{sl}_2)$ -submodule of V has type 1.*

Proof. Apply Lemma 3.7 to the $U_q(\mathfrak{sl}_2)$ -submodule in question. □

Definition 3.9. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. For $\lambda \in \mathbb{Z}$ define

$$V(\lambda) = \{v \in V \mid kv = q^\lambda v\}.$$

We call $V(\lambda)$ the λ -*weight space* of V . Note that $V(\lambda) \neq 0$ if and only if q^λ is an eigenvalue of k on V , and in this case $V(\lambda)$ is the corresponding eigenspace.

Lemma 3.10. (See [9, Section 2.2].) *Referring to Definition 3.9, for each weight space $V(\lambda)$ we have*

$$eV(\lambda) \subseteq V(\lambda + 2), \quad fV(\lambda) \subseteq V(\lambda - 2).$$

Lemma 3.11. *Each finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1 is a direct sum of its weight spaces.*

Proof. By Lemma 3.7 and Definition 3.9. □

By an *automorphism* of $U_q(\mathfrak{sl}_2)$ we mean an \mathbb{F} -algebra isomorphism from $U_q(\mathfrak{sl}_2)$ to $U_q(\mathfrak{sl}_2)$.

4 The Lusztig automorphism of $U_q(\mathfrak{sl}_2)$

In this section we recall the Lusztig automorphism of $U_q(\mathfrak{sl}_2)$. Our treatment follows [9, Chapter 8], more or less. The Lusztig automorphism has two versions, which we now define.

Definition 4.1. (See [9, Section 8.14].) The *Lusztig automorphism* L of $U_q(\mathfrak{sl}_2)$ sends

$$e \mapsto -fk, \quad f \mapsto -k^{-1}e, \quad k \mapsto k^{-1}.$$

The *Lusztig automorphism* L^\vee of $U_q(\mathfrak{sl}_2)$ sends

$$e \mapsto -kf, \quad f \mapsto -ek^{-1}, \quad k \mapsto k^{-1}.$$

The inverses of L and L^\vee are described as follows.

Lemma 4.2. *The automorphism L^{-1} sends*

$$e \mapsto -k^{-1}f, \quad f \mapsto -ek, \quad k \mapsto k^{-1}.$$

The automorphism $(L^\vee)^{-1}$ sends

$$e \mapsto -fk^{-1}, \quad f \mapsto -ke, \quad k \mapsto k^{-1}.$$

Proof. Routine using Definition 4.1. □

We now explain how L and L^\vee are related.

Lemma 4.3. *The following diagram commutes:*

$$\begin{array}{ccc} U_q(\mathfrak{sl}_2) & \xrightarrow{I} & U_q(\mathfrak{sl}_2) \\ L \downarrow & & \downarrow L^\vee \\ U_q(\mathfrak{sl}_2) & \xrightarrow{u \mapsto k u k^{-1}} & U_q(\mathfrak{sl}_2) \end{array}$$

Proof. Chase the $U_q(\mathfrak{sl}_2)$ generators $e, f, k^{\pm 1}$ around the diagram using Definition 4.1. □

Consider the Lusztig automorphisms L and L^\vee . Associated with L (resp. L^\vee) is a certain operator T (resp. T^\vee) called its Lusztig operator, that acts on each type 1 finite-dimensional $U_q(\mathfrak{sl}_2)$ -module in an \mathbb{F} -linear fashion. We now give the action.

Definition 4.4. (See [9, Section 8.2].) The Lusztig operators T and T^\vee act as follows. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. On each weight space $V(\lambda)$,

$$T = \sum_{\substack{a,b,c \in \mathbb{N} \\ b-a-c=\lambda}} \frac{e^a f^b e^c}{[a]_q! [b]_q! [c]_q!} (-1)^b q^{b-ac},$$

$$T^\vee = \sum_{\substack{a,b,c \in \mathbb{N} \\ a-b+c=\lambda}} \frac{f^a e^b f^c}{[a]_q! [b]_q! [c]_q!} (-1)^b q^{b-ac}.$$

Lemma 4.5. (See [9, Lemma 8.4].) *Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. Then T and T^\vee are invertible on V . On each weight space $V(\lambda)$,*

$$T^{-1} = \sum_{\substack{a,b,c \in \mathbb{N} \\ a-b+c=\lambda}} \frac{f^a e^b f^c}{[a]_q! [b]_q! [c]_q!} (-1)^b q^{ac-b},$$

$$(T^\vee)^{-1} = \sum_{\substack{a,b,c \in \mathbb{N} \\ b-a-c=\lambda}} \frac{e^a f^b e^c}{[a]_q! [b]_q! [c]_q!} (-1)^b q^{ac-b}.$$

We now describe how $T^{\pm 1}, (T^\vee)^{\pm 1}$ act.

Lemma 4.6. (See [9, Lemma 8.3].) *Refer to the basis $\{v_i\}_{i=0}^d$ of \mathbf{V}_d from Lemma 3.2. For $0 \leq i \leq d$,*

$$Tv_i = (-1)^{d-i} q^{(d-i)(i+1)} v_{d-i}, \quad T^\vee v_i = (-1)^i q^{i(d-i+1)} v_{d-i},$$

$$T^{-1}v_i = (-1)^i q^{i(i-d-1)} v_{d-i}, \quad (T^\vee)^{-1}v_i = (-1)^{d-i} q^{(i-d)(i+1)} v_{d-i}.$$

Lemma 4.7. (See [9, Section 8.2].) *Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. Then for $\lambda \in \mathbb{Z}$,*

$$TV(\lambda) = V(-\lambda), \quad T^\vee V(\lambda) = V(-\lambda).$$

The operators T and T^\vee are related in the following way.

Lemma 4.8. (See [9, Lemma 8.4].) *Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. On each weight space $V(\lambda)$,*

$$T = (-1)^\lambda q^\lambda T^\vee, \quad T^{-1} = (-1)^\lambda q^\lambda (T^\vee)^{-1}.$$

The Lusztig automorphism L (resp. L^\vee) is related to the Lusztig operator T (resp. T^\vee) in the following way.

Lemma 4.9. (See [9, Lemma 8.5].) *On each finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1,*

$$L(\xi) = T\xi T^{-1}, \quad L^\vee(\xi) = T^\vee \xi (T^\vee)^{-1} \quad \forall \xi \in U_q(\mathfrak{sl}_2).$$

5 The equitable presentation of $U_q(\mathfrak{sl}_2)$

In this section we recall the equitable presentation of $U_q(\mathfrak{sl}_2)$. For more information on this topic see [1, 3, 4, 6–8, 12, 14–19]. See also [2].

Lemma 5.1. (See [8, Theorem 2.1].) $U_q(\mathfrak{sl}_2)$ is isomorphic to the \mathbb{F} -algebra with generators $x, y^{\pm 1}, z$ and relations $yy^{-1} = 1, y^{-1}y = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1. \quad (5)$$

Given $0 \neq \theta \in \mathbb{F}$ and $t \in \mathbb{Z}$, an isomorphism sends

$$x \mapsto k^{-1} - k^{-1-t}eq^{1+t}(q - q^{-1})\theta^{-1}, \quad (6)$$

$$y^{\pm 1} \mapsto k^{\pm 1}, \quad (7)$$

$$z \mapsto k^{-1} + fk^tq^{-t}(q - q^{-1})\theta. \quad (8)$$

The inverse of this isomorphism sends

$$e \mapsto y^t(1 - yx)q^{-1-t}(q - q^{-1})^{-1}\theta,$$

$$f \mapsto (z - y^{-1})y^{-t}q^t(q - q^{-1})^{-1}\theta^{-1},$$

$$k^{\pm 1} \mapsto y^{\pm 1}.$$

Another isomorphism sends

$$x \mapsto k - k^{1+t}fq^{1+t}(q - q^{-1})\theta^{-1}, \quad (9)$$

$$y^{\pm 1} \mapsto k^{\mp 1}, \quad (10)$$

$$z \mapsto k + ek^{-t}q^{-t}(q - q^{-1})\theta. \quad (11)$$

The inverse of this isomorphism sends

$$e \mapsto (z - y^{-1})y^{-t}q^t(q - q^{-1})^{-1}\theta^{-1},$$

$$f \mapsto y^t(1 - yx)q^{-1-t}(q - q^{-1})^{-1}\theta,$$

$$k^{\pm 1} \mapsto y^{\mp 1}.$$

Proof. In each case, one checks that the maps in opposite direction are inverse \mathbb{F} -algebra homomorphisms, and hence \mathbb{F} -algebra isomorphisms. \square

Definition 5.2. Pick $0 \neq \theta \in \mathbb{F}$ and $t \in \mathbb{Z}$. Under the *primary identification* (resp. *secondary identification*) of type (θ, t) we identify the algebra $U_q(\mathfrak{sl}_2)$ and the algebra given in Lemma 5.1, via the first (resp. second) isomorphism given in Lemma 5.1.

The normalized Casimir element Λ looks as follows from the equitable point of view.

Lemma 5.3. Under every identification from Definition 5.2, Λ is equal to each of the following:

$$\begin{array}{ll} qx + q^{-1}y + qz - qxyz, & q^{-1}x + qy + q^{-1}z - q^{-1}zyx, \\ qy + q^{-1}z + qx - qyzx, & q^{-1}y + qz + q^{-1}x - q^{-1}xzy, \\ qz + q^{-1}x + qy - qzxy, & q^{-1}z + qx + q^{-1}y - q^{-1}yxz. \end{array}$$

Proof. Consider the expression for Λ given in (3). Write this expression in terms of x, y, z using any identification. Under a primary identification, $\Lambda = qx + q^{-1}y + qz - qxyz$. Under a secondary identification, $\Lambda = q^{-1}x + qy + q^{-1}z - q^{-1}zyx$. The six displayed expressions in the lemma statement are equal by [15, Lemma 2.15]. The result follows. \square

Rearranging (5) we obtain

$$q(1 - xy) = q^{-1}(1 - yx), \quad q(1 - yz) = q^{-1}(1 - zy), \quad q(1 - zx) = q^{-1}(1 - xz).$$

Definition 5.4. (See [15, Definition 3.1].) The elements ν_x, ν_y, ν_z of $U_q(\mathfrak{sl}_2)$ are defined as follows:

$$\begin{aligned} \nu_x &= q(1 - yz) = q^{-1}(1 - zy), \\ \nu_y &= q(1 - zx) = q^{-1}(1 - xz), \\ \nu_z &= q(1 - xy) = q^{-1}(1 - yx). \end{aligned}$$

Definition 5.5. (See [8, Definition 5.2].) The elements n_x, n_y, n_z of $U_q(\mathfrak{sl}_2)$ are defined as follows:

$$n_x = \frac{\nu_x}{q - q^{-1}}, \quad n_y = \frac{\nu_y}{q - q^{-1}}, \quad n_z = \frac{\nu_z}{q - q^{-1}}.$$

Lemma 5.6. Pick $0 \neq \theta \in \mathbb{F}$ and $t \in \mathbb{Z}$. Under the primary identification of type (θ, t) ,

$$\begin{aligned} e &= \theta q^{-t} y^t n_z, & f &= -\theta^{-1} q^{1+t} n_x y^{-1-t}, \\ n_z &= \theta^{-1} q^t k^{-t} e, & n_x &= -\theta q^{-1-t} f k^{1+t}. \end{aligned}$$

Under the secondary identification of type (θ, t) ,

$$\begin{aligned} f &= \theta q^{-t} y^t n_z, & e &= -\theta^{-1} q^{1+t} n_x y^{-1-t}, \\ n_z &= \theta^{-1} q^t k^t f, & n_x &= -\theta q^{-1-t} e k^{-1-t}. \end{aligned}$$

Proof. Use Definition 5.2 and Definitions 5.4, 5.5. \square

Lemma 5.7. (See [15, Lemma 6.4].) The \mathbb{F} -algebra $U_q(\mathfrak{sl}_2)$ is generated by $\nu_x, y^{\pm 1}, \nu_z$ and also by $n_x, y^{\pm 1}, n_z$. Moreover

$$x = y^{-1} - q^{-1} \nu_z y^{-1}, \quad z = y^{-1} - q^{-1} y^{-1} \nu_x.$$

Lemma 5.8. (See [8, Lemma 5.4].) The following hold in $U_q(\mathfrak{sl}_2)$:

$$\begin{aligned} x n_y &= q^2 n_y x, & y n_z &= q^2 n_z y, & z n_x &= q^2 n_x z, \\ x n_z &= q^{-2} n_z x, & y n_x &= q^{-2} n_x y, & z n_y &= q^{-2} n_y z. \end{aligned}$$

In [16] we described the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -modules from the equitable point of view. Below we summarize a few points.

Lemma 5.9. (See [16, Theorem 10.12].) *The $U_q(\mathfrak{sl}_2)$ -module $\mathbf{V}_{d,\varepsilon}$ has a basis $\{u_i\}_{i=0}^d$ with respect to which the matrices representing x, y, z are:*

$$\begin{aligned}
x : \quad & \varepsilon \begin{pmatrix} q^{-d} & q^d - q^{-d} & & & \mathbf{0} \\ & q^{2-d} & q^d - q^{2-d} & & \\ & & q^{4-d} & \cdot & \\ & & & \ddots & \\ & & & & q^d - q^{d-2} \\ \mathbf{0} & & & & & q^d \end{pmatrix}, \\
y : \quad & \varepsilon \operatorname{diag}(q^d, q^{d-2}, q^{d-4}, \dots, q^{-d}), \\
z : \quad & \varepsilon \begin{pmatrix} & q^{-d} & & & \mathbf{0} \\ q^{-d} - q^{2-d} & & q^{2-d} & & \\ & q^{-d} - q^{4-d} & q^{4-d} & & \\ & & \cdot & \cdot & \\ & & & \cdot & \\ \mathbf{0} & & & & q^{-d} - q^d & q^d \end{pmatrix}.
\end{aligned}$$

Lemma 5.10. *For each of x, y, z the action on $\mathbf{V}_{d,\varepsilon}$ is multiplicity-free, with eigenvalues $\{\varepsilon q^{d-2i} \mid 0 \leq i \leq d\}$.*

Proof. By Lemma 5.9. □

Lemma 5.11. (See [16, Theorem 11.7].) *Referring to Lemma 5.9, with respect to the basis $\{u_i\}_{i=0}^d$ the matrices representing n_x, n_z are*

$$\begin{aligned}
n_x : \quad & \begin{pmatrix} 0 & & & & \mathbf{0} \\ [1]_q & 0 & & & \\ & q^{-1}[2]_q & 0 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \\ \mathbf{0} & & & q^{1-d}[d]_q & 0 \end{pmatrix}, \\
n_z : \quad & \begin{pmatrix} 0 & -q^{d-1}[d]_q & & & \mathbf{0} \\ & 0 & -q^{d-2}[d-1]_q & & \\ & & 0 & \cdot & \\ & & & \cdot & \\ & & & & \cdot & -[1]_q \\ \mathbf{0} & & & & & 0 \end{pmatrix}.
\end{aligned}$$

Lemma 5.12. (See [16, Lemma 8.2].) *On the $U_q(\mathfrak{sl}_2)$ -module $\mathbf{V}_{d,\varepsilon}$, each of n_x^r, n_y^r, n_z^r is nonzero for $0 \leq r \leq d$, and*

$$n_x^{d+1} = 0, \quad n_y^{d+1} = 0, \quad n_z^{d+1} = 0.$$

From the equitable point of view we now consider finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules that are not necessarily irreducible.

Lemma 5.13. *Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. Then y acts on each weight space $V(\lambda)$ as a scalar multiple of the identity. The scalar is q^λ (under a primary identification) and $q^{-\lambda}$ (under a secondary second identification).*

Proof. By Definition 3.9 and (7), (10). \square

Lemma 5.14. *Each of n_x, n_y, n_z is nilpotent on finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules.*

Proof. Suppose the result is false, and let the finite-dimensional $U_q(\mathfrak{sl}_2)$ -module V be a counterexample with minimal dimension. By assumption $V \neq 0$. Also, V is not irreducible by Lemma 5.12 and the last assertion of Lemma 3.2. Therefore V contains a $U_q(\mathfrak{sl}_2)$ -submodule W such that $0 \neq W \neq V$. Since the counterexample V has minimal dimension, each of n_x, n_y, n_z is nilpotent on the $U_q(\mathfrak{sl}_2)$ -modules W and V/W . Consequently each of n_x, n_y, n_z is nilpotent on V , for a contradiction. The result follows. \square

By Lemma 5.1 the element y is invertible in $U_q(\mathfrak{sl}_2)$. By [8, Section 3] the elements x, z are not invertible in $U_q(\mathfrak{sl}_2)$. However by [8, Corollary 4.5] the elements x, z are invertible on nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules, provided that $\text{Char}(\mathbb{F}) \neq 2$. We now show that x, z are invertible on nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules, without any assumption about \mathbb{F} .

Lemma 5.15. *The elements x and z are invertible on nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules.*

Proof. The proof is similar to the proof of Lemma 5.14. Suppose the result is false, and let the nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -module V be a counterexample with minimal dimension. Then V is not irreducible by Lemma 5.10 and the last assertion of Lemma 3.2. Therefore V contains a $U_q(\mathfrak{sl}_2)$ -submodule W such that $0 \neq W \neq V$. Since the counterexample V has minimal dimension, x and z are invertible on the $U_q(\mathfrak{sl}_2)$ -modules W and V/W . Consequently x and z are invertible on V , for a contradiction. The result follows. \square

We now describe the Lusztig automorphisms L and L^\vee from the equitable point of view.

Proposition 5.16. *The following (i), (ii) hold for all $0 \neq \theta \in \mathbb{F}$ and $t \in \mathbb{Z}$.*

- (i) *Under the primary (resp. secondary) identification of type (θ, t) the automorphism L (resp. L^\vee) sends*

$$n_x \mapsto \theta^2 q^{-1} y^{-1} n_z y^{-1}, \quad y \mapsto y^{-1}, \quad n_z \mapsto \theta^{-2} q n_x$$

and L^{-1} (resp. $(L^\vee)^{-1}$) sends

$$n_x \mapsto \theta^2 q^{-1} n_z, \quad y \mapsto y^{-1}, \quad n_z \mapsto \theta^{-2} q y^{-1} n_x y^{-1}.$$

- (ii) *Under the primary (resp. secondary) identification of type (θ, t) the automorphism L^\vee (resp. L) sends*

$$n_x \mapsto \theta^2 q y^{-1} n_z y^{-1}, \quad y \mapsto y^{-1}, \quad n_z \mapsto \theta^{-2} q^{-1} n_x$$

and $(L^\vee)^{-1}$ (resp. L^{-1}) sends

$$n_x \mapsto \theta^2 q n_z, \quad y \mapsto y^{-1}, \quad n_z \mapsto \theta^{-2} q^{-1} y^{-1} n_x y^{-1}.$$

Proof. Use Definition 4.1 and Lemmas 4.2, 5.6. □

We point out two special cases of Proposition 5.16.

Corollary 5.17. *The following (i), (ii) hold for all $0 \neq \theta \in \mathbb{F}$ and $t \in \mathbb{Z}$.*

- (i) *Assume that $\theta^2 = q$. Under the primary (resp. secondary) identification of type (θ, t) the automorphism L (resp. L^\vee) sends*

$$n_x \mapsto y^{-1}n_z y^{-1}, \quad y \mapsto y^{-1}, \quad n_z \mapsto n_x$$

and L^{-1} (resp. $(L^\vee)^{-1}$) sends

$$n_x \mapsto n_z, \quad y \mapsto y^{-1}, \quad n_z \mapsto y^{-1}n_x y^{-1}.$$

- (ii) *Assume that $\theta^2 = q^{-1}$. Under the primary (resp. secondary) identification of type (θ, t) the automorphism L^\vee (resp. L) sends*

$$n_x \mapsto y^{-1}n_z y^{-1}, \quad y \mapsto y^{-1}, \quad n_z \mapsto n_x$$

and $(L^\vee)^{-1}$ (resp. L^{-1}) sends

$$n_x \mapsto n_z, \quad y \mapsto y^{-1}, \quad n_z \mapsto y^{-1}n_x y^{-1}.$$

Shortly we will describe the Lusztig operators T and T^\vee from the equitable point of view. In this description we will use the q -exponential function.

6 The q -exponential function

We will be discussing the elements n_x, n_y, n_z of $U_q(\mathfrak{sl}_2)$ from Definition 5.5. In this section we recall the q -exponential function, and investigate

$$\exp_q(n_x), \quad \exp_q(n_y), \quad \exp_q(n_z). \tag{12}$$

Definition 6.1. (See [13, p. 204].) Let V denote a vector space over \mathbb{F} with finite positive dimension. Let $\varphi \in \text{End}(V)$ be nilpotent. Define

$$\exp_q(\varphi) = \sum_{i \in \mathbb{N}} \frac{q^{\binom{i}{2}}}{[i]_q!} \varphi^i. \tag{13}$$

The following result is well known and readily verified.

Lemma 6.2. (See [13, p. 204].) *Referring to Definition 6.1, the map $\exp_q(\varphi)$ is invertible; its inverse is*

$$\exp_{q^{-1}}(-\varphi) = \sum_{i \in \mathbb{N}} \frac{(-1)^i q^{-\binom{i}{2}}}{[i]_q!} \varphi^i.$$

We will make use of the following identity.

Lemma 6.3. *Referring to Definition 6.1, we have*

$$\exp_q(q^2\varphi) (1 - (q^2 - 1)\varphi) = \exp_q(\varphi). \quad (14)$$

Proof. To verify (14), express each side as a power series in φ using (13), and for $i \in \mathbb{N}$ compare the coefficients of φ^i . \square

We now consider the objects (12). In view of Lemma 5.14, we interpret these objects as operators that act on nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules. As we investigate these operators we will display some results for $\exp_q(n_z)$; similar results hold for $\exp_q(n_x)$ and $\exp_q(n_y)$.

Lemma 6.4. *On nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules,*

- (i) $\exp_q(n_z)^{-1} \Lambda \exp_q(n_z) = \Lambda$;
- (ii) $\exp_q(n_z)^{-1} n_z \exp_q(n_z) = n_z$.

Proof. (i) The central element Λ commutes with n_z . Therefore Λ commutes with $\exp_q(n_z)$ in view of Definition 6.1.

(ii) By Definition 6.1. \square

Lemma 6.5. *On nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules,*

$$\exp_q(n_z)^{-1} y \exp_q(n_z) = x^{-1}.$$

Proof. Setting $\varphi = n_z$ in Lemma 6.3, we obtain $\exp_q(q^2 n_z) (1 - (q^2 - 1)n_z) = \exp_q(n_z)$. By Lemma 5.8 we have $yn_z y^{-1} = q^2 n_z$, so $y \exp_q(n_z) y^{-1} = \exp_q(q^2 n_z)$. By Definitions 5.4, 5.5 we have $n_z = q^{-1}(1 - yx)(q - q^{-1})^{-1}$, so $yx = 1 - (q^2 - 1)n_z$. By these comments $y \exp_q(n_z) x = \exp_q(n_z)$. The result follows. \square

Lemma 6.6. *For an integer $i \geq 1$,*

$$zn_z^i - n_z^i z = q^{1-i} [i]_q (n_z^{i-1} x - yn_z^{i-1}). \quad (15)$$

Proof. Using Lemma 5.3 and Definition 5.4 we obtain $z\nu_z = \Lambda - q^{-1}x - qy$ and $\nu_z z = \Lambda - qx - q^{-1}y$. Also $\nu_z = (q - q^{-1})n_z$ by Definition 5.5. By these comments $zn_z - n_z z = x - y$. Use this, Lemma 5.8, and induction on i to obtain (15). \square

Lemma 6.7. *On nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules,*

$$(y + z) \exp_q(n_z) = \exp_q(n_z) (x + z). \quad (16)$$

Proof. To verify (16), evaluate each side using Definition 6.1 and simplify the result using Lemma 6.6. \square

Lemma 6.8. *On nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules,*

$$\exp_q(n_z)^{-1} z \exp_q(n_z) = x - x^{-1} + z. \quad (17)$$

Proof. The left-hand side of (17) is equal to

$$\exp_q(n_z)^{-1}(y+z)\exp_q(n_z) - \exp_q(n_z)^{-1}y\exp_q(n_z),$$

which is equal to $x+z-x^{-1}$ by Lemmas 6.5, 6.7. \square

Lemma 6.9. *On nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules,*

$$\exp_q(n_z)^{-1}x\exp_q(n_z) = xyx.$$

Proof. In the equation $\exp_q(n_z)^{-1}n_z\exp_q(n_z) = n_z$, use $n_z = q(1-xy)(q-q^{-1})^{-1}$ to obtain $\exp_q(n_z)^{-1}(1-xy)\exp_q(n_z) = 1-xy$. Evaluate this equation using Lemma 6.5 to get the result. \square

Lemma 6.10. *On nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules,*

$$\exp_q(n_z)^{-1}n_x\exp_q(n_z) = x^{-1}n_yx^{-1}. \quad (18)$$

Proof. To verify (18), eliminate n_x, n_y using

$$n_x = q(1-yz)(q-q^{-1})^{-1}, \quad n_y = q(1-zx)(q-q^{-1})^{-1},$$

and evaluate the result using Lemmas 6.5, 6.8. \square

Lemma 6.11. *On nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules,*

$$\exp_q(n_z)^{-1}n_y\exp_q(n_z) = \frac{\Lambda x}{q-q^{-1}} + n_y - \frac{q+q^{-1}}{q-q^{-1}}x^2 + xn_zx. \quad (19)$$

Proof. Using Lemma 5.3 and Definition 5.4 we obtain $\nu_y y = \Lambda - qz - q^{-1}x$. In this equation, multiply each side on the left and right by $\exp_q(n_z)^{-1}$ and $\exp_q(n_z)$, respectively, and evaluate the result using Definitions 5.4, 5.5 and Lemmas 6.5, 6.8, 6.9. \square

7 Rotators

In this section we discuss the rotators for a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. We comment on the history. In [8] a rotator was constructed using the q -exponential function. In [16, Section 16] we found the matrices that represent this rotator with respect to various bases for the underlying vector space. In [18, Section 22] we investigated the rotator concept in a more general setting. In the present section we will follow [8] more or less, adopting a different point of view, and giving new proofs that we find more illuminating.

Definition 7.1. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. By a *rotator* on V we mean an invertible $R \in \text{End}(V)$ such that on V ,

$$R^{-1}xR = y, \quad R^{-1}yR = z, \quad R^{-1}zR = x. \quad (20)$$

Lemma 7.2. *Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. Then there exists a rotator on V .*

Proof. By Definition 3.6 we may assume that the $U_q(\mathfrak{sl}_2)$ -module V is irreducible. Define x', y', z' in $\text{End}(V)$ as follows:

element	x'	y'	z'
action on V	y	z	x

The map y' is invertible by Lemma 5.15. The maps x', y', z' satisfy the defining relations for $U_q(\mathfrak{sl}_2)$; therefore V becomes a $U_q(\mathfrak{sl}_2)$ -module such that

element	x	y	z
action on V	x'	y'	z'

The new $U_q(\mathfrak{sl}_2)$ -module V is irreducible by construction, and type 1 by Lemma 5.10. Therefore the new $U_q(\mathfrak{sl}_2)$ -module V is isomorphic to the original $U_q(\mathfrak{sl}_2)$ -module V . Let $R \in \text{End}(V)$ denote an isomorphism of $U_q(\mathfrak{sl}_2)$ -modules from the new $U_q(\mathfrak{sl}_2)$ -module V to the original $U_q(\mathfrak{sl}_2)$ -module V . By construction R is invertible and satisfies (20). Therefore R is a rotator on V . \square

Later in the paper we will construct an operator that acts as a rotator on the finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules of type 1. For the time being, we focus on the irreducible case.

Lemma 7.3. *Let R denote a rotator on the irreducible $U_q(\mathfrak{sl}_2)$ -module $V = \mathbf{V}_d$. Then for $R' \in \text{End}(V)$ the following are equivalent:*

- (i) R' is a rotator on V ;
- (ii) there exists $0 \neq \alpha \in \mathbb{F}$ such that $R' = \alpha R$.

Proof. (i) \Rightarrow (ii) The composition $R'R^{-1}$ commutes with the actions x, y, z on V . By this and Lemma 5.9, there exists $0 \neq \alpha \in \mathbb{F}$ such that $R'R^{-1} = \alpha I$. Consequently $R' = \alpha R$.

(ii) \Rightarrow (i) Clear. \square

Definition 7.4. Consider the irreducible $U_q(\mathfrak{sl}_2)$ -module $V = \mathbf{V}_d$. Define X, Y, Z in $\text{End}(V)$ as follows. By Lemma 5.10, each of x, y, z is multiplicity-free on V with eigenvalues $\{q^{d-2i} \mid 0 \leq i \leq d\}$. For $0 \leq i \leq d$, X (resp. Y) (resp. Z) acts on the (q^{d-2i}) -eigenspace of x (resp. y) (resp. z) as $q^{2i(d-i)}$ times the identity. By construction X, Y, Z are invertible.

We have some comments about Definition 7.4. Since the scalars $\{q^{d-2i}\}_{i=0}^d$ are mutually distinct, there exists a polynomial $G = G_d$ in one variable, that has all coefficients in \mathbb{F} and degree at most d , such that

$$G(q^{d-2i}) = q^{2i(d-i)} \quad (0 \leq i \leq d).$$

In the above line, replace i by $d - i$ to obtain

$$G(q^{2i-d}) = q^{2i(d-i)} \quad (0 \leq i \leq d).$$

By construction the following hold on the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_d :

$$X = G(x) = G(x^{-1}), \quad (21)$$

$$Y = G(y) = G(y^{-1}), \quad (22)$$

$$Z = G(z) = G(z^{-1}). \quad (23)$$

We call G the *standard polynomial* for \mathbf{V}_d . We now give some results for Y ; similar results hold for X, Z .

Lemma 7.5. *On each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1, $Yy = yY$ and*

$$Y^{-1}n_xY = y^{-1}n_xy^{-1}, \quad Yn_xY^{-1} = yn_xy, \quad (24)$$

$$Y^{-1}n_zY = yn_zy, \quad Yn_zY^{-1} = y^{-1}n_zy^{-1}. \quad (25)$$

Proof. Let V denote the module in question. So $V = \mathbf{V}_d$, where d is the diameter of V . To verify the given equations, consider the matrices that represent each side with respect to the basis $\{u_i\}_{i=0}^d$ of V from Lemma 5.9. With respect to $\{u_i\}_{i=0}^d$ the matrix representing y is given in Lemma 5.9, and the matrices representing n_x, n_z are given in Lemma 5.11. With respect to $\{u_i\}_{i=0}^d$ the matrix representing Y is diagonal, with (i, i) -entry $q^{2i(d-i)}$ for $0 \leq i \leq d$. The results follow from these comments after a brief calculation. \square

Recall the primary and secondary identifications from Definition 5.2.

Lemma 7.6. *Let V denote a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1. Under any identification, Y acts on each weight space $V(\lambda)$ as a scalar multiple of the identity. The scalar is $q^{(d^2-\lambda^2)/2}$, where d is the diameter of V .*

Proof. By Lemma 5.13 and Definition 7.4, along with

$$2i(d-i) = \frac{d^2 - (d-2i)^2}{2} \quad (0 \leq i \leq d).$$

\square

Definition 7.7. Let V denote a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1. Define $\Omega \in \text{End}(V)$ by

$$\Omega = \exp_q(n_x) Y \exp_q(n_z), \quad (26)$$

where Y is from Definition 7.4.

Proposition 7.8. *Let V denote a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1. Then the map Ω from Definition 7.7 is a rotator on V .*

Proof. We verify that Ω satisfies the conditions of Definition 7.1. Note that Ω is invertible. We show that $\Omega^{-1}x\Omega = y$. We have

$$\begin{aligned} \exp_q(n_x)^{-1} x \exp_q(n_x) &= x + y - y^{-1} \\ &= y - q^{-1}\nu_z y^{-1} \end{aligned}$$

and

$$\begin{aligned} Y^{-1}(y - q^{-1}\nu_z y^{-1})Y &= y - q^{-1}Y^{-1}\nu_z Y y^{-1} \\ &= y(1 - q^{-1}\nu_z) \end{aligned}$$

and

$$\begin{aligned} \exp_q(n_z)^{-1} y(1 - q^{-1}\nu_z) \exp_q(n_z) &= x^{-1}(1 - q^{-1}\nu_z) \\ &= y. \end{aligned}$$

By these comments and (26) we find $\Omega^{-1}x\Omega = y$. Next we show that $\Omega^{-1}y\Omega = z$. We have

$$\begin{aligned} \exp_q(n_x)^{-1} y \exp_q(n_x) &= yzy \\ &= (1 - q^{-1}\nu_x)y \end{aligned}$$

and

$$\begin{aligned} Y^{-1}(1 - q^{-1}\nu_x)yY &= y - q^{-1}Y^{-1}\nu_x Y y \\ &= y - q^{-1}y^{-1}\nu_x \\ &= y - y^{-1} + z \end{aligned}$$

and

$$\begin{aligned} \exp_q(n_z)^{-1} (y - y^{-1} + z) \exp_q(n_z) &= x^{-1} - x + x - x^{-1} + z \\ &= z. \end{aligned}$$

By these comments and (26) we find $\Omega^{-1}y\Omega = z$. Next we show that $\Omega^{-1}z\Omega = x$. We have

$$\begin{aligned} \exp_q(n_x)^{-1} z \exp_q(n_x) &= y^{-1}, \\ Y^{-1}y^{-1}Y &= y^{-1}, \\ \exp_q(n_z)^{-1} y^{-1} \exp_q(n_z) &= x. \end{aligned}$$

By these comments and (26) we find $\Omega^{-1}z\Omega = x$. We have shown that Ω is a rotator on V . \square

Definition 7.9. Let V denote a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1. By the *standard rotator on V* we mean the rotator Ω from Definition 7.7.

Lemma 7.10. *On each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1, the standard rotator Ω satisfies*

$$\Omega^{-1}n_x\Omega = n_y, \quad \Omega^{-1}n_y\Omega = n_z, \quad \Omega^{-1}n_z\Omega = n_x \quad (27)$$

and also

$$\Omega^{-1} \exp_q(n_x) \Omega = \exp_q(n_y), \quad (28)$$

$$\Omega^{-1} \exp_q(n_y) \Omega = \exp_q(n_z), \quad (29)$$

$$\Omega^{-1} \exp_q(n_z) \Omega = \exp_q(n_x). \quad (30)$$

Proof. Use Definition 7.1. □

Lemma 7.11. *On each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1, the standard rotator Ω satisfies*

$$\Omega^{-1}X\Omega = Y, \quad \Omega^{-1}Y\Omega = Z, \quad \Omega^{-1}Z\Omega = X. \quad (31)$$

Proof. Let V denote the $U_q(\mathfrak{sl}_2)$ -module in question, and let G denote the standard polynomial for V . We show that $\Omega^{-1}X\Omega = Y$ holds on V . The equation $\Omega^{-1}x\Omega = y$ holds on V . By this and (21), (22) we find that on V ,

$$\Omega^{-1}X\Omega = \Omega^{-1}G(x)\Omega = G(\Omega^{-1}x\Omega) = G(y) = Y.$$

The remaining equations in (31) are similarly obtained. □

Proposition 7.12. *On every finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1, the standard rotator Ω is equal to each of*

$$\exp_q(n_x) Y \exp_q(n_z), \quad \exp_q(n_y) Z \exp_q(n_x), \quad \exp_q(n_z) X \exp_q(n_y). \quad (32)$$

Proof. In the equation (26), multiply each side on the left and right by Ω^{-1} and Ω , respectively. Evaluate the result using (28)–(30) and Lemma 7.11. □

Note 7.13. The operator \mathcal{R} from [16, Definition 17.8] is the same thing as our Ω^{-1} . Also, the operator called Ω in [8, Definition 7.4] is the same thing as our $\Omega q^{-d^2/2}$ (if d is even) and $\Omega q^{(1-d^2)/2}$ (if d is odd). Here d denotes the diameter of the $U_q(\mathfrak{sl}_2)$ -module in question.

Lemma 7.14. *On each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1, the map Ω^3 acts as a scalar multiple of the identity. The scalar is $(-1)^d q^{d(d-1)}$, where d is the diameter of the module.*

Proof. By [16, Lemma 16.5] or [8, Corollary 8.5], together with Note 7.13. □

We mention a result for later use.

Lemma 7.15. *On each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1,*

$$\exp_q(n_z)^{-1} Y \exp_q(n_z) = X.$$

Proof. Let V denote the $U_q(\mathfrak{sl}_2)$ -module in question, and let G denote the standard polynomial for V . By Lemma 6.5 the following holds on V :

$$\exp_q(n_z)^{-1} y \exp_q(n_z) = x^{-1}.$$

Therefore on V ,

$$\exp_q(n_z)^{-1} G(y) \exp_q(n_z) = G(x^{-1}).$$

The result follows in view of (21), (22). □

In Proposition 7.12 we gave three formulae for the standard rotator. We now give additional formulae for this rotator.

Proposition 7.16. *On every finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1, the standard rotator Ω is equal to each of*

$$\exp_q(n_x) \exp_q(n_z) X, \quad \exp_q(n_y) \exp_q(n_x) Y, \quad \exp_q(n_z) \exp_q(n_y) Z, \quad (33)$$

$$Z \exp_q(n_x) \exp_q(n_z), \quad X \exp_q(n_y) \exp_q(n_x), \quad Y \exp_q(n_z) \exp_q(n_y). \quad (34)$$

Proof. By Proposition 7.12 and Lemma 7.15. \square

8 The maps τ_x, τ_y, τ_z and the Lusztig operators T, T^\vee

Let V denote a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1, with standard rotator Ω . In this section we use Ω to define three elements in $\text{End}(V)$, denoted τ_x, τ_y, τ_z . We discuss how τ_x, τ_y, τ_z are related to the maps X, Y, Z from Definition 7.4. We then show how τ_y is related to the Lusztig operators T and T^\vee .

Definition 8.1. Let V denote a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1. Define τ_x, τ_y, τ_z in $\text{End}(V)$ by

$$\tau_x = \exp_q(n_y) \Omega, \quad \tau_y = \exp_q(n_z) \Omega, \quad \tau_z = \exp_q(n_x) \Omega.$$

We have some comments about Definition 8.1.

Lemma 8.2. *On each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1, the maps τ_x, τ_y, τ_z are invertible.*

Lemma 8.3. *On each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1,*

$$\tau_x = \Omega \exp_q(n_z), \quad \tau_y = \Omega \exp_q(n_x), \quad \tau_z = \Omega \exp_q(n_y).$$

Proof. By (28)–(30) and Definition 8.1. \square

Next we give some results for τ_y ; similar results hold for τ_x, τ_z .

Lemma 8.4. *On each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1,*

- (i) $\tau_y^{-1} n_x \tau_y = y^{-1} n_z y^{-1}$;
- (ii) $\tau_y^{-1} y \tau_y = y^{-1}$;
- (iii) $\tau_y^{-1} n_z \tau_y = n_x$.

Proof. Let V denote the $U_q(\mathfrak{sl}_2)$ -module in question.

(i) Using Lemma 6.10 we find that on V ,

$$\tau_y^{-1} n_x \tau_y = \Omega^{-1} \exp_q(n_z)^{-1} n_x \exp_q(n_z) \Omega = \Omega^{-1} x^{-1} n_y x^{-1} \Omega = y^{-1} n_z y^{-1}.$$

(ii) Using Lemma 6.5 we find that on V ,

$$\tau_y^{-1} y \tau_y = \Omega^{-1} \exp_q(n_z)^{-1} y \exp_q(n_z) \Omega = \Omega^{-1} x^{-1} \Omega = y^{-1}.$$

(iii) Using Lemma 6.4(ii) we find that on V ,

$$\tau_y^{-1} n_z \tau_y = \Omega^{-1} \exp_q(n_z)^{-1} n_z \exp_q(n_z) \Omega = \Omega^{-1} n_z \Omega = n_x.$$

□

Lemma 8.5. *On each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1,*

$$X = \Omega^3 \tau_x^{-2}, \quad Y = \Omega^3 \tau_y^{-2}, \quad Z = \Omega^3 \tau_z^{-2}. \quad (35)$$

Proof. We verify the middle equation. We have

$$Y \tau_y^2 = Y \exp_q(n_z) \Omega \exp_q(n_z) \Omega. \quad (36)$$

Consider the right-hand side of (36). We have $Y \exp_q(n_z) = \exp_q(n_z) X$ by Lemma 7.15, and $\Omega \exp_q(n_z) = \exp_q(n_y) \Omega$ by (29). Also $\exp_q(n_z) X \exp_q(n_y) = \Omega$ by Lemma 7.12. By these comments, the right-hand side of (36) is equal to Ω^3 . The result follows. □

Recall the primary and secondary identifications from Definition 5.2.

Lemma 8.6. *Let V denote a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module of type 1. Under any identification, $\tau_y V(\lambda) = V(-\lambda)$ for all $\lambda \in \mathbb{Z}$.*

Proof. Use Lemma 5.13 and Lemma 8.4(ii). □

Proposition 8.7. *Pick $0 \neq \theta \in \mathbb{F}$ and $t \in \mathbb{Z}$.*

- (i) *Assume that $\theta^2 = q$. Under the primary identification of type (θ, t) the following holds on the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_d :*

$$\tau_y = (-1)^d \theta^{d^2} T^{-1}. \quad (37)$$

- (ii) *Assume that $\theta^2 = q$. Under the secondary identification of type (θ, t) the following holds on the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_d :*

$$\tau_y = (-1)^d \theta^{d^2} (T^\vee)^{-1}. \quad (38)$$

- (iii) *Assume that $\theta^2 = q^{-1}$. Under the primary identification of type (θ, t) the following holds on the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_d :*

$$\tau_y = \theta^{-d^2} (T^\vee)^{-1}. \quad (39)$$

- (iv) *Assume that $\theta^2 = q^{-1}$. Under the secondary identification of type (θ, t) the following holds on the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_d :*

$$\tau_y = \theta^{-d^2} T^{-1}. \quad (40)$$

Proof. For the time being, assume any identification. On $V = \mathbf{V}_d$,

$$\begin{aligned}
\tau_y &= \Omega \exp_q(n_x) \\
&= \exp_q(n_x) Y \exp_q(n_z) \exp_q(n_x) \\
&= \exp_q(n_x) Y \exp_q(n_z) Y^{-1} Y \exp_q(n_x) Y^{-1} Y \\
&= \exp_q(n_x) \exp_q(Y n_z Y^{-1}) \exp_q(Y n_x Y^{-1}) Y \\
&= \exp_q(n_x) \exp_q(y^{-1} n_z y^{-1}) \exp_q(y n_x y) Y.
\end{aligned}$$

Consider the action of τ_y on a weight space $V(\lambda)$. By construction and Lemma 7.6 we find that on $V(\lambda)$,

$$\tau_y = \sum_{a,b,c \in \mathbb{N}} \frac{q^{\binom{a}{2}} q^{\binom{b}{2}} q^{\binom{c}{2}}}{[a]_q! [b]_q! [c]_q!} n_x^a (y^{-1} n_z y^{-1})^b (y n_x y)^c q^{(d^2 - \lambda^2)/2}. \quad (41)$$

(i) We show that (37) holds on V . Pick $a, b, c \in \mathbb{N}$ and consider the corresponding summand in (41). We now write this summand in terms of e, f, k . Using the primary identification of type (θ, t) in Lemma 5.6, along with $ke = q^2 ek$ and $kf = q^{-2} fk$, we obtain

$$\begin{aligned}
n_x^a &= (-\theta q^{-1-t} f k^{1+t})^a = (-1)^a \theta^a q^{-a^2(1+t)} f^a k^{a(1+t)}, \\
(y^{-1} n_z y^{-1})^b &= (\theta^{-1} q^t k^{-1-t} e k^{-1})^b = \theta^{-b} q^{-b^2(2+t)} e^b k^{-b(2+t)}, \\
(y n_x y)^c &= (-\theta q^{-1-t} k f k^{2+t})^c = (-1)^c \theta^c q^{-c^2(3+t)} f^c k^{c(3+t)}, \\
f^a k^{a(1+t)} e^b k^{-b(2+t)} f^c k^{c(3+t)} &= f^a e^b f^c k^{a(1+t)-b(2+t)+c(3+t)} q^{2ab(1+t)-2ac(1+t)+2bc(2+t)}.
\end{aligned}$$

Observe that on $V(\lambda)$,

$$k^{a(1+t)-b(2+t)+c(3+t)} = q^{\lambda(a(1+t)-b(2+t)+c(3+t))}.$$

By the above comments, for a, b, c the corresponding summand in (41) acts on $V(\lambda)$ as a scalar multiple of $f^a e^b f^c$. By Lemma 3.10 we have $f^a e^b f^c V(\lambda) \subseteq V(\mu)$, where $\lambda - \mu = 2(a - b + c)$. Note that $\mu = -\lambda$ if and only if $a - b + c = \lambda$. By this and Lemma 8.6, the equation (41) remains valid if we restrict the sum to those $a, b, c \in \mathbb{N}$ such that $a - b + c = \lambda$. Evaluating (41) using the above discussion we find that on $V(\lambda)$,

$$\tau_y = (-1)^d \theta^{d^2} \sum_{\substack{a,b,c \in \mathbb{N} \\ a-b+c=\lambda}} \frac{f^a e^b f^c}{[a]_q! [b]_q! [c]_q!} (-1)^b q^{ac-b}. \quad (42)$$

By (42) and Lemma 4.5, we find that $\tau_y = (-1)^d \theta^{d^2} T^{-1}$ on $V(\lambda)$. Since V is spanned by its weight spaces, $\tau_y = (-1)^d \theta^{d^2} T^{-1}$ on V .

(ii) We show that (38) holds on V . Recall that τ_y acts on each weight space $V(\lambda)$ as in (41). Pick $a, b, c \in \mathbb{N}$ and consider the corresponding summand in (41). We now write this summand in terms of e, f, k . Using the secondary identification of type (θ, t) in Lemma 5.6, along with $ke = q^2 ek$ and $kf = q^{-2} fk$, we obtain

$$\begin{aligned}
n_x^a &= (-\theta q^{-1-t} e k^{-1-t})^a = (-1)^a \theta^a q^{-a^2(1+t)} e^a k^{-a(1+t)}, \\
(y^{-1} n_z y^{-1})^b &= (\theta^{-1} q^t k^{1+t} f k)^b = \theta^{-b} q^{-b^2(2+t)} f^b k^{b(2+t)}, \\
(y n_x y)^c &= (-\theta q^{-1-t} k^{-1} e k^{-2-t})^c = (-1)^c \theta^c q^{-c^2(3+t)} e^c k^{-c(3+t)}, \\
e^a k^{-a(1+t)} f^b k^{b(2+t)} e^c k^{-c(3+t)} &= e^a f^b e^c k^{-a(1+t)+b(2+t)-c(3+t)} q^{2ab(1+t)-2ac(1+t)+2bc(2+t)}.
\end{aligned}$$

On $V(\lambda)$,

$$k^{-a(1+t)+b(2+t)-c(3+t)} = q^{\lambda(-a(1+t)+b(2+t)-c(3+t))}.$$

This time around, (41) remains valid if we restrict the sum to those $a, b, c \in \mathbb{N}$ such that $b - a - c = \lambda$. Evaluating (41) using these comments we find that on $V(\lambda)$,

$$\tau_y = (-1)^d \theta^{d^2} \sum_{\substack{a, b, c \in \mathbb{N} \\ b - a - c = \lambda}} \frac{e^a f^b e^c}{[a]_q! [b]_q! [c]_q!} (-1)^b q^{ac-b}. \quad (43)$$

By (43) and Lemma 4.5, we find that $\tau_y = (-1)^d \theta^{d^2} (T^\vee)^{-1}$ on $V(\lambda)$. Since V is spanned by its weight spaces, $\tau_y = (-1)^d \theta^{d^2} (T^\vee)^{-1}$ on V .

(iii) Proceeding as in the proof of (i) above, we find that on each weight space $V(\lambda)$,

$$\tau_y = (-1)^\lambda q^{-\lambda} \theta^{-d^2} \sum_{\substack{a, b, c \in \mathbb{N} \\ a - b + c = \lambda}} \frac{f^a e^b f^c}{[a]_q! [b]_q! [c]_q!} (-1)^b q^{ac-b}. \quad (44)$$

Evaluate (44) using Lemmas 4.5, 4.8.

(iv) Proceeding as in the proof of (ii) above, we find that on each weight space $V(\lambda)$,

$$\tau_y = (-1)^\lambda q^\lambda \theta^{-d^2} \sum_{\substack{a, b, c \in \mathbb{N} \\ b - a - c = \lambda}} \frac{e^a f^b e^c}{[a]_q! [b]_q! [c]_q!} (-1)^b q^{ac-b}. \quad (45)$$

Evaluate (45) using Lemmas 4.5, 4.8. □

9 The rotator \mathfrak{R} and the Lusztig operators T, T^\vee

In this section we introduce an operator \mathfrak{R} that acts as a rotator on each finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. We show how \mathfrak{R} is related to the Lusztig operators T, T^\vee .

Until the end of this section, the following notation and assumptions are in effect. Assume that \mathbb{F} contains the square root of q . Pick $\theta \in \mathbb{F}$ such that $\theta^2 = q$ or $\theta^2 = q^{-1}$. Pick $t \in \mathbb{Z}$. Assume the primary or secondary identification of type (θ, t) , as in Definition 5.2. Define

$$q^{1/2} = \begin{cases} -\theta & \text{if } \theta^2 = q; \\ \theta^{-1} & \text{if } \theta^2 = q^{-1}. \end{cases} \quad (46)$$

Lemma 9.1. *For $d \in \mathbb{N}$,*

$$q^{d^2/2} = \begin{cases} (-1)^d \theta^{d^2} & \text{if } \theta^2 = q; \\ \theta^{-d^2} & \text{if } \theta^2 = q^{-1}. \end{cases}$$

Proof. The integers d and d^2 have the same parity, so $(-1)^d = (-1)^{d^2}$. The result follows from this and (46). □

Next we define an operator Υ , that acts on each type 1 finite-dimensional $U_q(\mathfrak{sl}_2)$ -module in an \mathbb{F} -linear fashion. We now give the action.

Definition 9.2. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. Then Υ acts on each weight space $V(\lambda)$ as a scalar multiple of the identity. The scalar is $q^{-\lambda^2/2}$, where $q^{1/2}$ is from (46).

We have two comments about Υ .

Lemma 9.3. *Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. Then each $U_q(\mathfrak{sl}_2)$ -submodule of V is Υ -invariant.*

Proof. Let W denote the $U_q(\mathfrak{sl}_2)$ -submodule in question. By Lemmas 3.8, 3.11 we see that W is spanned by eigenvectors for k . By Definition 9.2, these vectors are eigenvectors for Υ . Therefore W is Υ -invariant. \square

Lemma 9.4. *On the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_d ,*

$$Y = q^{d^2/2} \Upsilon.$$

Proof. By Lemma 7.6 and Definition 9.2. \square

We now define an operator \mathfrak{R} that acts on each type 1 finite-dimensional $U_q(\mathfrak{sl}_2)$ -module in an \mathbb{F} -linear fashion.

Definition 9.5. On each finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1,

$$\mathfrak{R} = \exp_q(n_x) \Upsilon \exp_q(n_z), \quad (47)$$

where Υ is from Definition 9.2.

Lemma 9.6. *On the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_d ,*

$$\mathfrak{R} = q^{-d^2/2} \Omega,$$

where $q^{1/2}$ is from (46).

Proof. Compare (26) and (47) using Lemma 9.4. \square

Lemma 9.7. *Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. Then each $U_q(\mathfrak{sl}_2)$ -submodule of V is \mathfrak{R} -invariant.*

Proof. Let W denote the $U_q(\mathfrak{sl}_2)$ -submodule in question. By assumption W is invariant under n_x and n_z . Therefore W is invariant under $\exp_q(n_x)$ and $\exp_q(n_z)$. By Lemma 9.3, W is invariant under Υ . By these comments and Definition 9.5, W is invariant under \mathfrak{R} . \square

Lemma 9.8. *The operator \mathfrak{R} acts as a rotator on each finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1.*

Proof. Let V denote the $U_q(\mathfrak{sl}_2)$ -module in question. By Definition 3.6, V is a direct sum of irreducible $U_q(\mathfrak{sl}_2)$ -submodules that have type 1. Each summand W is \mathfrak{R} -invariant by Lemma 9.7. By Lemma 9.6, \mathfrak{R} acts on W as a scalar multiple of Ω . Now by Lemma 7.3 and Proposition 7.8, \mathfrak{R} acts on W as a rotator. Consequently \mathfrak{R} acts on V as a rotator. \square

Recall from Definition 5.2 the primary and secondary identification of type (θ, t) .

Theorem 9.9. *The rotator \mathfrak{R} is related to the Lusztig operators T, T^\vee according to the table below:*

	primary ident. of type (θ, t)	secondary ident. of type (θ, t)
$\theta^2 = q$	$T^{-1} = \exp_q(n_z) \mathfrak{R}$	$(T^\vee)^{-1} = \exp_q(n_z) \mathfrak{R}$
$\theta^2 = q^{-1}$	$(T^\vee)^{-1} = \exp_q(n_z) \mathfrak{R}$	$T^{-1} = \exp_q(n_z) \mathfrak{R}$

Proof. First assume $\theta^2 = q$ and the primary identification of type (θ, t) . Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type 1. We show that $T^{-1} = \exp_q(n_z) \mathfrak{R}$ on V . By Definition 3.6 and Lemma 9.7, we may assume without loss that the $U_q(\mathfrak{sl}_2)$ -module V is irreducible. Let d denote the diameter of V . On V ,

$$\begin{aligned}
T^{-1} &= (-1)^d \theta^{-d^2} \tau_y && \text{by Proposition 8.7(i)} \\
&= q^{-d^2/2} \tau_y && \text{by Lemma 9.1} \\
&= q^{-d^2/2} \exp_q(n_z) \Omega && \text{by Definition 8.1} \\
&= \exp_q(n_z) \mathfrak{R} && \text{by Lemma 9.6.}
\end{aligned}$$

For the other cases the proof is similar. \square

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